UNIT-2: Second-Order Linear Differential Equations with Variable Coefficients

Second-Order Linear Differential Equations with Variable Coefficients: Solution by Method of Undetermined Coefficients, By Known Integral, Removal of First Derivative, Change of Independent Variable and Variation of Parameters.

SECOND ORDER LINEAR DIFFERENTIAL EQUATION WITH VARIABLE COEFFICIENTS

Second Order Diffrential Equation with Variable coefficent(or Linear diff. Equation of second order: The Standerd form of *Second Order Diffrential Equation with Variable coefficent is* $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=R(x)$

Where P(x), Q(x) and R(x) are function of *x* alone or may be constants.

EQUATION WHOSE ONE SOLUTION IS KNOWN

Method-I : Steps for solution:

1. Let $y=u.v$ is complete solution (Where u or v is the one part of the solution).
2. Substitute the values of y , y' and y'' in given eq..
3. Now we get Second order diff. eq. with constant coefficients in terms of v.
4. Solve this reduced equation for v and find the value of Complete solution $y=u.v$.

1. Solve $x\frac{d^2y}{dx^2}-(2x-1)\frac{dy}{dx}+(x-1)y=e^x$ given that $y = e^x$ is an integral.**[Dec.03,07,08, Feb.2010, June 2010, Nov. 2018]**

[Hint : Take $y=u.v=e^x.v$, $\frac{dy}{dx}=e^x v+e^x\frac{dv}{dx}$, $\frac{d^2y}{dx^2}=e^x\frac{d^2v}{dx^2}+2e^x\frac{dv}{dx}+e^x v$, we get $\frac{d^2v}{dx^2}+\frac{1}{x}\frac{dv}{dx}=\frac{1}{x}$, put $\frac{dv}{dx}=t$, Then we get $\frac{dt}{dx}+\frac{1}{x}t=\frac{1}{x}$, which is linear eq. , its sol $t(x)=x+c$ *or* $x\frac{dv}{dx}-x=c$, on solving we get $v=x+c\log x+c_1$ and complete sol. $y=u.v=e^x.v=e^x(x+c\log x+c_1)$ **]**

2. Solve $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0$ given that $x+\frac{1}{x}$ is one integral **[Dec.2002,Jan .2006, June 2011,Dec.2014]**

3. Solve $\sin^2 x\frac{d^2y}{dx^2}=2y$,given that $y=\cot x$ is a solution. **[.Jan. 2007]**

4. Find C.F. of the differential equation $(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+y=0$ if $y=x$ is one part of the solution**[June 2016,**

To find One integral of C.F. by inspection: Method-I : If $1+P/a+Q/a^2=0$ **then** $y=e^{ax}$ ***is*** **one part of the solution (C.F.).**

If $1+P+Q=0$ then $y=e^x$ *is* one part of the solution (C.F.).

If $1-P+Q=0$ then $y=e^{-x}$ *is* one part of the solution (C.F.).

Method-I : If $m(m-1)+Pmx+Qx^2=0$ **then** $y=x^m$ **is one part of the solution (C.F.).**

If $P+Qx=0$ then $y=x$ is one part of the solution.

If $2+2Px+Qx^2=0$ then $y=x^2$ is one part of the solution (C.F.).

For Solution sue method –I

1. Write one part of C.F. of the differential equation $(3-x)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+y=0$ **[June 2015]**

2. Solve $x^2\frac{d^2y}{dx^2}-2x(1+x)\frac{dy}{dx}+2(1+x)y=x^3$

3. Solve $\frac{d^2y}{dx^2}-\cot x\frac{dy}{dx}-(1-\cot x)y=e^x\sin x$ **[.Sept. 2009, Dec. 2014 June 17]**

4. Solve $x\frac{d^2y}{dx^2}-\frac{dy}{dx}+(1-x)y=x^2e^{-x}$ given that $y=e^x$ is an integral.

Method –II : NORMAL FORM , Removal of First Derivative [or Change of Dependent Variable]

1. Change the given equation in standerd form $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=R(x)$,and find *P(x), Q(x) and R(x).*

2. Let $u=e^{-\frac{1}{2}\int pdx}$

3. Let *y*= *u.v* is complete solution , where *u* is the one part of the solution , finding in step-2

4. Substitute the value of *y , y'* and *y''* in given equation , We get Normal equation

$$\frac{d^2v}{dx^2}+Iv=\frac{R}{u}, \text{Where } I=Q-\frac{1}{2}\frac{dP}{dx}-\frac{1}{4}P^2$$

6. Solve $x^2\frac{d^2y}{dx^2}-2(x^2+x)\frac{dy}{dx}+(x^2+2x+2)y=0$

Hint : $u=e^{-\frac{1}{2}\int Pdx}=e^{\int(1+\frac{1}{x})dx}=xe^x$, $I=Q-\frac{1}{2}\frac{dP}{dx}-\frac{1}{4}P^2=0$ and Normal Eq. $\frac{d^2v}{dx^2}+Iv=\frac{R}{u}\Rightarrow\frac{d^2v}{dx^2}=0$

7. Solve the differential equation $\frac{d^2y}{dx^2}-2\tan x\frac{dy}{dx}-5y=0$ by deducting it in normal form. **[June 18]**

Hint : $u=e^{-\frac{1}{2}\int Pdx}=e^{\int\tan xdx}=\sec x$, $I=Q-\frac{1}{2}\frac{dP}{dx}-\frac{1}{4}P^2=-4$ and Normal Eq. $\frac{d^2v}{dx^2}+Iv=\frac{R}{u}\Rightarrow\frac{d^2v}{dx^2}-4v=0$

8. Solve $\frac{d^2y}{dx^2}-2\tan x\frac{dy}{dx}+5y=e^x\sec x$ **[.June. 2010,June 2015]**

Hint : $u=e^{-\frac{1}{2}\int Pdx}=e^{\int\tan xdx}=\sec x$, $I=Q-\frac{1}{2}\frac{dP}{dx}-\frac{1}{4}P^2=6$ and Normal Eq. $\frac{d^2v}{dx^2}+Iv=\frac{R}{u}\Rightarrow\frac{d^2v}{dx^2}+6v=e^x$

9. Solve $\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+(4x^2-1)y=-3e^{x^2}\sin 2x$

10. Using method of removal of first derivative , solve the equation $\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+(x^2+1)y=x^3+3x$ **[June 17]**

METHOD OF CHANGING THE INDEPENDENT VARIABLE

1. Change the given equation in standard form $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=R(x)$,and find *P(x), Q(x) and R(x).*
2. Choose z such that $(\frac{dz}{dx})^2=Q$ (Q is taken with out –ve singn and surd)
3. Find $P_1=\frac{[(\frac{d^2z}{dx^2})+P\frac{dz}{dx}]}{(dz/dx)^2}, Q_1=\frac{Q}{(dz/dx)^2}=c_2=1 \text{ and } R_1=\frac{R}{(dz/dx)^2}$ Substitute these values in Transformed equation $\frac{d^2y}{dz^2}+P_1(x)\frac{dy}{dz}+Q_1(x)y=R_1(x)$
4. Solve this transformed equation by previous methods.

11. Solve $\frac{d^2y}{dx^2}-(1+4e^x)\frac{dy}{dx}+3e^{2x}y=2e^{2(x+e^x)}$, [**.Jan. 2006]**.

12. Solve $\cos x\frac{d^2y}{dx^2}+\sin x\frac{dy}{dx}-(2\cos^3 x)y=2\cos^5 x$ **[.June. 2003, Dec.2006]**

Hint :change the eq. in standard form $\frac{d^2y}{dx^2}+\tan x\frac{dy}{dx}-(2\cos 2x)y=2\cos^4 x$, Choose z such that $(\frac{dz}{dx})^2=Q=\cos^2 x\Rightarrow\frac{dz}{dx}=\cos x\Rightarrow z=\sin x$ and $\frac{d^2z}{dx^2}=-\sin x$, Reduced Eq. $\frac{d^2y}{dz^2}-2y=2\cos^2 x=2(1-z^2)$

13. Solve $(1+x^2)^2\frac{d^2y}{dx^2}+2x(1+x^2)\frac{dy}{dx}+4y=0$ **[.June. 2012]**

METHOD OF VARRIATION OF PARAMETERS

This method is used to find the complete solution, without finding the particular solution.

1. Find the C.F. of the equation
2. Let Complete solution is *y= A u(x) + B v(x)*, where *u(x)* and *v(x)* are the parts of *C.F.*

Where A and B can be determined by $A=\int\frac{Ru}{\begin{vmatrix}u & v\\ u' & v'\end{vmatrix}}dx+c_1 \text{ and } B=-\int\frac{Rv}{\begin{vmatrix}u & v\\ u' & v'\end{vmatrix}}dx+c_2$

14. Solve by the method of variation of parameters $(D^2+a^2)y=\sec ax$

Ans. $y=(c_1\cos ax+c_2\sin ax)+\frac{\cos ax}{a^2}\log(\cos ax)+\frac{x\sin ax}{a}$ **[June 2012]**

15. Solve by the method of variation of parameters $(D^2+4)y=\sec 2x$

Ans. $y=(c_1\cos 2x+c_2\sin 2x)+\frac{\cos 2x}{4}\log(\cos 2x)+\frac{x\sin 2x}{2}$ ***[RGPV* Dec 2007, *Dec. 2011]***

16. Solve by the method of variation of parameters $\frac{d^2y}{dx^2}+y=tx$ **[Nov. 2018]**

17. Solve by the method of variation of parameters $(D^2+4)y = 4\tan 2x$
Ans $y = c_1\cos 2x + c_2\sin 2x - \cos 2x\log(\sec 2x + \tan 2x) - \sin 2x\cos 2x$ **[June16 (CBCS)]**

18. Apply the method of variation of parameters to solve $\frac{d^2y}{dx^2} + a^2y = \operatorname{cosec} ax$

19. Apply the method of variation of parameters to solve $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$ **Dec. 2010, June. 2011, June 17]**

20. Apply the method of variation of parameters to solve $\frac{d^2y}{dx^2} + y = x$ [**June. 2015]**

21. Solve $x^2\frac{d^2y}{dx^2} - 2x(1+x)\frac{dy}{dx} + 2(1+x)y = x^3$ [**June 2007]**

22. Solve by the method of variation of parameters $(D^2-2D)y = e^x\sin x$ **Ans.:** $y = c_1 + c_2e^{2x} - \frac{1}{2}e^x\sin x$

23. Apply the method of variation of parameters to solve $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$

24. Apply the method of variation of parameters to solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = e^x\tan x$ [**Jan. 2002]**

25. Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \frac{e^{3x}}{x^2}$ **Ans.** $y = (c_1 + c_2x)e^{3x} - (1+\log x)e^{3x}$ **[Dec. 04, 06,07, June. 08, Feb. 10,June 18]**

26. Solve $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2e^x$ [**Sept. 2009]**

27. Solve $x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = 8x^3\sin x^2$

Series Solution of Differential equations:

1. **When x=0 is an ordinary point (i.e. at x=0 function is analytic)**
   Step-I *let Series solution is* $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n + \ldots$
   Step-II find y, dy/dx, d^2y/dx^2 and put in question
   Step-III Equate to 0 , the coefficient of $x^0, x^1, x^2, \ldots$and solve these relations, and get the values of constants $a_1, a_2, \ldots\ldots a_n$, and then find the general solution by equation of step 1.
2. **When x=0 is a Regular singular point *i.e.* not an ordinary point (*i.e.* at x=0 function is not analytic)[Frobenius Method]**
   Step-I *let Series solution is* $y = \sum_{r=0}^{\infty} a_r x^{m+r} = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n + \ldots$
   Step-II find y, dy/dx, d^2y/dx^2 and put in question.
   Step-III Put the lowest power of x =0 and find the *indicial equation* , and find the roots of indicial equation (i.e. different values of *m)*
   Step-IV If (i) roots of indicial equation are distinct and not differ by an integer then Complete solution $y = c_1(y)m = m_1 + c_2(y)m = m_2$